Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(01)00015-6

J. Appl. Maths Mechs, Vol. 65, No. 1, pp. 139–144, 2001

0021-8928/01/\$—see front matter

ASYMPTOTIC ANALYSIS OF WAVE FIELDS WHEN THERE IS PARTIAL IMPULSE DELAMINATION OF THE MEDIA†

A. A. ZOLOTAREV

Rostov-on-Don

(*Received* 6 *December* 1999)

A method of solving transient wave problems with mixed boundary conditions for multilayered media [1–3] is generalized to problems in which the continuity breaks down. Unlike existing results [1, 4, 5], obtained for the case of the propagation of only harmonic perturbations from the initial instant of time, the space–time structure of the wave fields in the case of pulsed generation modes is investigated by an asymptotic analysis of the solution of a system of Wiener–Hopf type functional equations. The conditions for weak wave effects to arise for transient waves, due to the layered structure of semi-infinite media, are analysed.

Consider a two-layer packet of elastic media of constant thickness h ($h = h_1 + h_2$), bounded in the direction of the horizontal plane-parallel boundaries. We will investigate the nature of the wave fields excited at the initial instant of time $t = 0$ by a pulse breakdown in the continuity in the half-plane D at the interface of the layers (delamination)

$$\mathbf{L}_j \cdot \mathbf{v}_j = \rho_j \frac{\partial^2}{\partial t^2}\mathbf{v}_j; \quad |\mathbf{x}| < \infty, \quad -h_1 \leqslant z \leqslant h_2 \tag{1}$$

$$\mathbf{L}_j = (\lambda_j + \mu_j)\,\mathrm{grad\,div} + \mu_j\,\mathrm{rot\,rot}$$

$$\mathbf{v}_j(\mathbf{x}, t) = (u_j, w_j), \quad \mathbf{p}_j(\mathbf{x}, t) = (\sigma_j, \tau_j), \quad j = 1, 2$$

$$\mathbf{x} = (x, z), \quad D = \{\mathbf{x}: |x > 0, \ z = 0\}$$

$$z = 0: \ \mathbf{v}_2 - \mathbf{v}_1 = \begin{cases} \Delta\mathbf{v}, & \mathbf{x} \in D \\ 0, & \mathbf{x} \notin D \end{cases}, \ \mathbf{p}_2 - \mathbf{p}_1 = \begin{cases} \Delta\mathbf{p}, & \mathbf{x} \in D \\ 0, & \mathbf{x} \notin D \end{cases}$$

$$z = -h_1: \ w_1 = 0, \ \tau_1 = 0; \ z = h_2: \ \mathbf{p}_2 = \mathbf{f}(x, t) \tag{2}$$

$$t = 0: \ \mathbf{v}_j = 0, \ \frac{\partial}{\partial t}\mathbf{v}_j = 0 \tag{3}$$

Here (x, z) is a Cartesian rectangular system of coordinates with origin at the interface of the media $z = 0$, ρ_j, λ_j, μ_j, and h_j are the density, Lamé parameters and the thickness of the jth layer, and $\mathbf{p}_j$, $\mathbf{v}_j$, $\Delta\mathbf{p}$ and $\Delta\mathbf{v}$ are the components of the stress tensor (σ_j and τ_j are the normal and tangential stresses), and the displacement vector and their jumps, respectively. We assume that the material obeys Hooke's law, while the stresses $\mathbf{p}_j$ on the edges of the breakdown in continuity in the region D are known (and consequently, also the jumps $\Delta\mathbf{p}$ in (2) and the external forces $\mathbf{f}$). We will use the zero initial conditions (3).

The method of Fourier and Laplace integral transformations with respect to the variables x and t with parameters α and s enables us to reduce problem (1)–(3) to a system of Wiener–Hopf type functional equations

$$\begin{gathered} K\Delta W^+ = S^+ + S^- - K_0(T^+ + T^-) + \mathbf{M}\cdot\mathbf{F} \\ G\Delta U^+ - K\Delta W^+ = G_0(T^+ + T^-) + \mathbf{N}\cdot\mathbf{F} \\ \alpha \in E, \quad \mathrm{Re}\, s \geqslant \delta_0 > 0 \end{gathered} \tag{4}$$

†*Prikl. Mat. Mekh.* Vol. 65, No. 1, pp. 141–147, 2001.

where E is a band of regularity in the complex plane α, common to all the functions in (4), containing the real axis, δ_0 is the abscissa of convergence of the Laplace transformation, and $S^{\pm}$, $T^{\pm}$, $\Delta W^{\pm}$, $\Delta U^{\pm}$ are the transforms of the functions σ, τ, Δw, Δu in the interface plane of the layers ($z = 0$). The superscript plus (minus) corresponds to the functions indicated, defined for $x > 0$ ($x < 0$), and everywhere henceforth denotes the regularity of their transforms in the upper half-plane or lower half-plane ($E \cup \{\alpha$: |Im $\alpha < 0\}$) as a function of the complex variable α.

For uniquely solvable initial-boundary-value problem (1)–(3), Green's functions (4) in transform space are unique, analytical and even with respect to the set of variables α and s with asymptotic behaviour

$$K, G = O(|\alpha|), \quad K_0, G_0 = O(1); \quad |\alpha| \to \infty, \quad s = \text{const}$$

For a two-layer packet of media, when there are no surface forces ($\mathbf{F} = 0$), the specific form of these functions was determined previously [4], but in view of the complexity of the expressions they are not reproduced here. Below, when constructing the solution we will only present the structure of the most necessary of these.

As is well known, outside the general regularity band E, Green's functions have a denumerable set of zeros and poles, which are the wave numbers defining the dispersion properties of the composite medium considered. In particular, for these we will introduce the notation

$$\alpha = \alpha_n^{\pm}(s), \quad \alpha = \eta_m^{\pm}(s), \quad K(\alpha_n^{\pm}, s) = K^{-1}(\eta_m^{\pm}, s) = 0 \tag{5}$$
$$\text{Im}\{\alpha_n^+, \eta_m^+\} > 0, \quad \text{Im}\{\alpha_n^-, \eta_m^-\} < 0, \quad \text{Re}\, s \geqslant \delta_0 > 0$$

To fix our ideas, the wave numbers will henceforth be assumed to be numbered in order of increase in their moduli.

The function factorization method, generalized to the case of non-stationary problems [3, 4], enables us to present the solution of system (4) for transforms of the required jumps in displacements in the region of the breakdown, and then, using the latter, we can also determine the wave fields in the whole waveguide. For example, in notation (4) and (5) we obtain the following expressions for the transformants of the stresses on the interface of the layers outside the defect

$$T^-(\alpha, s) = L^-(\alpha, s)\sum_m \frac{T_m^-}{\alpha - \eta_m^+}, \quad S^-(\alpha, s) = K^-(\alpha, s)\sum_m \frac{S_n^-}{\alpha - \alpha_n^+}$$
$$L^-(\alpha, s) = \frac{G^-(\alpha, s)}{G_0^-(\alpha, s)} \tag{6}$$
$$L^-(\eta_m^+, s) = 0, \quad (L^-(\beta_l^+, s))^{-1} = 0, \quad \text{Im}\,\beta_l^+ > 0$$
$$T_m^- = -\frac{\Delta W^+(\eta_m^+, s)}{G^-(\eta_m^+, s)G_0^+(\eta_m^+, s)(K^{-1}(\eta_m^+, s))'}$$
$$S_n^- = \frac{T^-(\alpha_n^+, s)K_0(\alpha_n^+, s) - S^+(\alpha_n^+, s)}{(K^-(\alpha_n^+, s))'}$$
$$(K^-(\alpha_n^+, s))' = \frac{\partial}{\partial\alpha}(K^-(\alpha, s))$$

Here the functions K, G, K_0 and G_0 are factorized in the form of a product (for example, $K(\alpha, s) = K^+(\alpha, s)K^+(\alpha, s)$), while the unknowns $\Delta W^+(\eta_m^+, s)$ satisfy an infinite linear system (whence they can also be determined, for example, by the reduction method [3, 4] or by the method of successive approximations [5])

$$\mathbf{A} \cdot \Delta\mathbf{W} = \mathbf{B} \tag{7}$$
$$\mathbf{A} = \{a_{ml}\}, \quad \Delta\mathbf{W} = \{\Delta W^+(\eta_m^+, s)\}, \quad \mathbf{B} = \{b_m\}, \quad m, n = 1, 2, \dots$$

Results (6) were obtained on the assumption that there are no shear loads on the discontinuity, while the normal loads are equal and opposite, i.e. $\{T^+, \Delta T^+, \Delta S^+\} = 0$. The latter assumptions do not reduce the generality of the discussion of the method of solution and the further qualitative analysis of the results.

Inverse Fourier and Laplace transformations, applied to the transforms of the solution obtained, enable us to determine the representations of the physical fields in integral form. Thus, for the stresses on the boundary of the interface of the media (6) we have

$$\mathbf{p}(x,t)=\frac{1}{4i\pi^2}\int_{\delta-i\infty}^{\delta+i\infty}e^{st}\,ds\int_{-\infty}^{+\infty}\mathbf{P}^-(\alpha,s)e^{-i\alpha x}d\alpha \tag{8}$$

$$\mathbf{P}^-(\alpha,s)=\{S^-(\alpha,s),T^-(\alpha,s)\};\quad x<0,\quad z=0,\quad \delta\geqslant\delta_0>0$$

The quantities S^- and T^- are given in (6), and in view of the continuity of the wave fields outside the region D where the continuity breaks down we use the notation

$$\mathbf{p}(x,t)=\mathbf{p}_j(\mathbf{x},t)|_{z=0}=\{\sigma(x,t),\tau(x,t)\},\quad j=1,2;\quad x<0$$

It can be shown that in relation (8) the inversion operators are commutative for a fairly wide class of perturbing factors, for example, $\mathbf{p}_j(\mathbf{x},t)\in C_{\{(\mathbf{x},t):|D\cap(t\geqslant 0)\}}$ are continuous and decreasing at infinity with respect to each of the variables separately ($|x|\to\infty$, $t\to\infty$) or decreasing with respect to x and monochromatic with respect to time t.

We will first evaluate the Fourier integral in relation (8), closing the contour of integration in the upper half-plane α, where the only poles of the integrands are $\eta_m^+(s)$ and $\beta_m^+(s)$ respectively. Using the theory of residues, we obtain for the stresses

$$\mathbf{p}(x,t)=\sum_m\mathbf{q}_m(x,t),\quad x<0,\quad z=0 \tag{9}$$

$$\mathbf{q}_m(x,t)=\frac{1}{2\pi}\int_{\delta-i\infty}^{\delta+i\infty}\mathbf{Q}_m(s)\exp(-ix\boldsymbol{\theta}_m+st\mathbf{I})\,ds \tag{10}$$

$$\mathbf{Q}_m(s)=\{S_m(s),T_m(s)\},\quad \mathbf{I}=\{1,1\}$$

$$\boldsymbol{\theta}_m\equiv\{\theta_{jm}\}_{j=1,2}=\{\eta_m^+,\beta_m^+\}$$

$$S_m(s)=\frac{1}{(K^{-1}(\eta_m^+,s))'}\sum_n\frac{S_n^-}{\alpha_n^+-\eta_m^+}$$

$$T_m(s)=\frac{1}{(L^{-1}(\beta_m^+,s))'}\sum_l\frac{T_l^-}{\beta_m^+-\eta_l^+}$$

For impulse delamination modes the Laplace transform of the components of the known dynamic forces in the region of the break has no singularities in the half-plane $\operatorname{Re}s\geqslant 0$. Consequently, we can put $\delta=0$, and integral (10) then reduces to integration along the imaginary axis which, putting $s=-i\omega$ leads to the more convenient expression

$$\mathbf{q}_m(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{Q}_m(-i\omega)\exp(i\boldsymbol{\Phi}_m(\omega,\gamma))d\omega \tag{11}$$

$$\boldsymbol{\Phi}_m(\omega,\gamma)=\{\eta_m^+(-i\omega)-\gamma\omega,\ \beta_m^+(-i\omega)-\gamma\omega\}$$

$$\boldsymbol{\Phi}_m(\omega,\gamma)\equiv\{\Phi_{jm}\}_{j=1,2}=\boldsymbol{\theta}_m(-i\omega)-\gamma\omega\mathbf{I}$$

$$\gamma=t/|x|$$

Here $\mathbf{q}_m=\{\sigma_m,\tau_m\}$ are the contributions of the corresponding modes with wave numbers η_m^+, β_m^+ in solution (9), describing wave fields for the stresses at the interface of the layers outside their delamination region D.

We will carry out an asymptotic analysis of integrals (11). For fixed $\omega=\text{const}$ the finite number of wave numbers is real-valued. They determine the fundamental modal composition of the transient wave packets, the amplitude factors of which, as will be shown below, decrease in a power fashion as $|x|,t\to\infty$. The remaining modes with numbers $m>M$, corresponding to complex-valued wave numbers, make exponentially decaying contributions and will henceforth be ignored in expressions (9) and (11).

For steady harmonic oscillations, the dispersion properties of a wide class of semi-infinite multilayered media (when $s = -i\omega$, $\omega > 0$) have been widely investigated [1]. An analysis of non stationary problems for these media, using a generalization of the factorization method and representing the solution in the form of expansions in eigenfunctions, requires that the dispersion law α_m^+, η_m^+, β_m^+ should be established over the whole range of complex-valued s. The necessary analytical extension of the dispersion relations into this region is made by establishing the natural physical requirement that the group velocities of the modes considered should be positive [3]

$$C_{jm} = (\theta'_{jm})^{-1} > 0, \quad -\infty < \omega < +\infty \tag{12}$$

$$\mathbf{C}_m^{-1} \equiv \{C_{jm}^{-1}(\omega)\}_{j=1,2} = \boldsymbol{\theta}'_m = \partial\boldsymbol{\theta}_m(-i\omega)/\partial\omega$$

As a characteristic example, we show in the figure, in dimensionless form, a number of first dispersion relations of real-valued (for $s = -i\omega$) wave numbers $\alpha = \alpha_n^+$ ($n = 1, 2, 3, 4$) and the group velocities $C = C_n$ corresponding to them. One can change from dimensionless quantities by using the characteristic values of the length $h = h_1 + h_2$, the velocities of the longitudinal waves $\upsilon = \max \upsilon_{j1}$ ($\upsilon_{j1} = \sqrt{(\lambda_j - \mu_j)/\rho_j}$) and the density $\rho = \max \rho_j$. The results are given for dimensionless thicknesses of the layers $H_j = h_j/h$ ($H_1 = 0.3$, $H_2 = 0.7$ and $H = H_1 + H_2$), relative densities $\rho_j^* = \rho_j/\rho$ ($\rho^* = 0.7$, $\rho_2^* = 1$), frequency $\varkappa$, and velocities of the longitudinal waves υ_{jp} and transverse waves υ_{js}, related to the dimensional quantities as follows:

$$\varkappa = \frac{\omega h}{\upsilon_{21}}, \quad \upsilon_{jp} = \frac{\upsilon_{j1}}{\upsilon_{21}} = \left[\frac{\rho_2(\lambda_j + \mu_j)}{\rho_j(\lambda_2 + \mu_2)}\right]^{1/2}$$

$$\upsilon_{js} = \frac{\upsilon_{j2}}{\upsilon_{21}} = \left[\frac{\rho_2 \mu_j}{\rho_j(\lambda_2 + \mu_2)}\right]^{1/2}; \quad j = 1, 2$$

Using localization principles [6, 7] the main contributions to the value of integral (11) are made by the poles of the integrands and the singular points of the phase functions. By virtue of the pulsed nature of the delamination of the waveguide with respect to time, the amplitude functions in (11) can have poles only in the left half-plane Im $s < 0$, which do not reach the integration contour. In view of the their exponential attenuation with time their contributions are ignored. Hence, the main contribution to the asymptotic behaviour of integral (11) is made by the real-valued stationary phase points

$$\omega = \pm\omega_{mn} \quad (n = 1, 2, \ldots, N; \quad \partial\Phi_m(\pm\omega_{mn}, \gamma)/\partial\omega = 0) \tag{13}$$

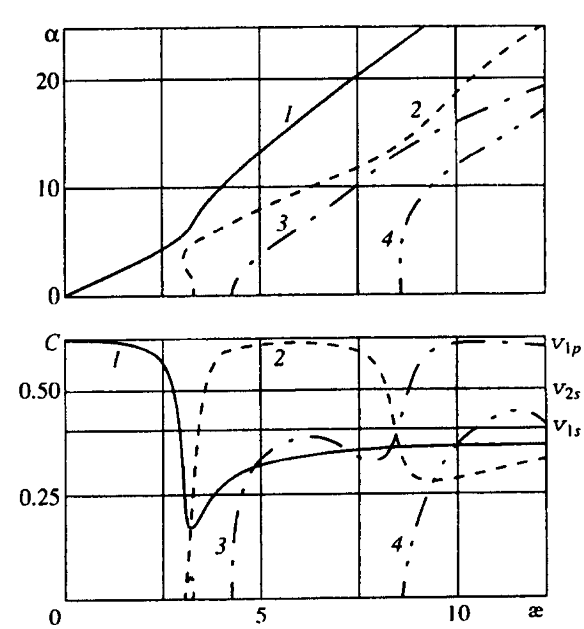

Fig. 1

In the case of isolated single stationary points (13), using the stationary-phase method [6], we obtain from (11)

$$\mathbf{q}_m(x,t) = \mathbf{q}_m^- - \mathbf{q}_m^+ + O(|t|^{-1}) \tag{14}$$

$$\mathbf{q}_m^{\pm} = \sum_{n=1}^{N} \frac{\mathbf{Y}_m(\pm\omega_{mn})}{|x|^{1/2}} \exp\left(\pm i|x|\mathbf{\Phi}_m(\omega_{mn},\gamma) \pm \frac{i\pi}{4}\chi_{mn}\mathbf{I}\right)$$

$$\mathbf{Y}_m(\omega) \equiv \{Y_{jm}(\omega)\}_{j=1,2} = \frac{U(\gamma)}{(2\pi)^{1/2}}\left\{\frac{S_m(i\omega)}{|\theta_{1m}''|^{1/2}},\ \frac{T_m(i\omega)}{|\theta_{2m}''|^{1/2}}\right\}$$

$$\theta_{jm}'' = \frac{\partial^2\theta_{jm}(\pm i\omega_{mn})}{\partial\omega^2} \neq 0, \quad \mathbf{\theta}_m'(\pm i\omega_{mn}) - \gamma\mathbf{I} = 0$$

$$\chi_{mn} = \operatorname{sign}\theta_{jm}'', \quad U(\gamma) = \begin{cases} 1, & (x,t) \in D_n \\ 0, & (x,t) \notin D_n \end{cases}$$

$$D_n(x,t):\ \{C_{jmn}^- t < |x| < C_{jmn}^+ t\}$$

$$\gamma \equiv t/|x| = \text{const}, \quad x,t \to \infty$$

For the extremum values of the group velocities we have used the following notation above

$$C_{jmn}^- = \min_{k=n-1,n}(C_{jmk}^*), \quad C_{jmn}^+ = \max_{k=n-1,n}(C_{jmk}^*)$$

$$C_{jmk}^* = C_{jm}(\omega_{mk}^*)$$

where ω_{mn}^* are the boundary points of the interval Ω_n: $\{\omega_{mn-1}^* \leqslant \omega_{mn} \leqslant \omega_{mn}^*\}$ are the regions in which the n-th stationary point ω_{mn} (13) exits. Hence, in representation (14) Ω_n defines the space-time region D_n, filled with the n-th wave packet of the m-th mode for the j-th component of the wave field. The boundary values ω_{mn}^* in the interval Ω_n (with the exception, possibly, of ω_{m0}^* and $\omega_{mN}^* = \infty$) are double stationary points of the corresponding phase functions

$$\partial\mathbf{\Phi}_m(\pm\omega_{mn}^*,\gamma^*)/\partial\omega = \partial^2\mathbf{\Phi}_m(\pm\omega_{mn}^*,\gamma^*)/\partial\omega^2 = 0$$

Hence, as $\omega_{mn} \to \pm\omega_{mn}^*$ (which occurs as $|x| \to C_{jmn}^* \cdot t$) integral (11) has an asymptotic representation which differs from (14) and can be described by an asymptotic from using Airy functions [6, 7] or by the equivalent expansion [3]

$$\mathbf{q}_m(x,t) = \mathbf{q}_m^- - \mathbf{q}_m^+ + O(t^{-2/3}) \tag{15}$$

$$\mathbf{q}_m^{\pm} = \sum_{n=1}^{N} \frac{\mathbf{B}_m(\pm\omega_{mn}^*)}{|x|^{1/3}} \exp(\pm i|x|\mathbf{\Phi}_m(\omega_{mn}^*,\gamma))$$

$$\mathbf{B}_m(\omega) \equiv \{B_{jm}(\omega)\} = \frac{\Gamma(1/3)}{\pi\cdot 2^{1/3}}\left\{\frac{\chi_{mn}S_m(i\omega)}{|\theta_{1m}'''|^{1/3}},\ \frac{\chi_{mn}T_m(i\omega)}{|\theta_{2m}'''|^{1/3}}\right\}$$

$$j = 1,2$$

$$\mathbf{\theta}_m'(\pm i\omega_{mn}^*) - \gamma^*\mathbf{I} = 0, \quad \mathbf{\theta}_m''(\pm i\omega_{mn}^*) = 0$$

$$\theta_{jm}''' = \frac{\partial^3\theta_{jm}(\pm i\omega_{mn}^*)}{\partial\omega^3} \neq 0, \quad \chi_{mn} = U(\gamma)\operatorname{sign}\theta_{jm}'''$$

$$U(\gamma) = \begin{cases} 1 & (x,t) \in D_n^* \\ 0, & (x,t) \notin D_n^* \end{cases}, \quad D_n^*:\ \left\{\left||x| - C_{jmn}^* t\right| < \varepsilon \ll 1\right\}$$

$$t \to \infty, \quad |x| \to C_{jmn}^* t \quad (\gamma \to \gamma^*)$$

It can be seen from (9), (14) and (15) that the wave field generated by a time-pulsed form of partial delamination of the media, in space-time regions D_n, is the superposition of wave packets, the amplitudes of which decrease with distance and attenuate with time ($|x| \to \infty$, $t \to \infty$) as $|x|^{-1/2}$, $t^{-1/2}$, respectively. These waves reach their greatest amplitudes in the frontal zones D_n (in the zone of the front $|x| \to C_{jmn}^{-}t$ and rear $|x| \to C_{jmn}^{-}t$ of each packet). Here the amplitude factors decrease and attenuate more weakly, proportional to $|x|^{-1/3}$, $t^{-1/3}$.

It should be noted that the formation of these wave structures (with a slower decrease in amplitudes) is a characteristic feature of multilayered media only and can be interpreted as the occurrence of "weak" wave effects for transient waves, generated by pulsed sources.

REFERENCES

1. VOROVICH, I. I. and BABESHKO, V. A., *Dynamic Mixed Problems of the Theory of Elasticity Non-classical Regions*. Nauka, Moscow, 1979.
2. PORUCHIKOV, V. B., *Methods of the Dynamic Theory of Elasticity*. Nauka, Moscow, 1986.
3. ZOLOTAREV, A. A., An approach to solving the integral equations of initial-boundary-value problems for multilayered media. *Izv. Akad. Nauk SSSR. MTT*, 1990, **6**, 30–35.
4. ZAITSEV, I. A. and ZOLOTAREV, A. A., The space-time structure of the wave field in a two-layer medium with boundary friction. *Prikl. Mat. Mekh.*, 1993, **57**, 1, 172–179.
5. ANDREICHIKOV, I. I. and ZOLOTAREV, A. A., The diffraction of waves by a thin inclusion in a two-layer waveguide. *Akust. Zh.*, 1993, **39**, 5, 773–781.
6. FEDORYUK, M. V., *The Method of Steepest Descent*. Nauka, Moscow, 1977.
7. FOCKE, J., Asimptotische Entwicklungen mittels der Methode der stationären Phase. *Ber. Verhandl. Sächs. Akad. Wiss. Math-naturwiss*. Kl. 1954, **101**, 1–18.

Translated by R.C.G.